

DECOMPOSING SIMPLE PERMUTATIONS, WITH ENUMERATIVE CONSEQUENCES

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Received June 8, 2006

We prove that every sufficiently long simple permutation contains two long almost disjoint simple subsequences, and then we show how this result has enumerative consequences. For example, it implies that, for any r , the number of permutations with at most r copies of 132 has an algebraic generating function (this was previously proved, constructively, by Bóna and (independently) Mansour and Vainshtein).

1. Statement of theorem

Simplicity, under a variety of names¹, has been studied for a wide range of combinatorial objects. Our main result concerns simple permutations; possible analogues for other contexts are discussed in the conclusion. An *interval* in the permutation π is a set of contiguous indices $I = [a, b]$ such that the set of values $\pi(I) = \{\pi(i) : i \in I\}$ also forms an interval of natural numbers. Every permutation π of $[n] = \{1, 2, \dots, n\}$ has intervals of size 0, 1, and n ; π is said to be *simple* if it has no other intervals. [Figure 1](#) shows the plots of two simple permutations. Intervals of permutations are interesting in their own right and have applications to biomathematics; see Corteel, Louchard, and Pemantle [10], where among other results it is proved that the number of simple permutations of $[n]$ is asymptotic to $n!/e^2$. More precise asymptotics are given by Albert, Atkinson, and Klazar [2].

Mathematics Subject Classification (2000): 05D99, 05A15, 06A07

* Supported by a Royal Society Dorothy Hodgkin Research Fellowship.

† Supported by EPSRC grant GR/S53503/01.

¹ Two synonyms for simplicity are primality and indecomposability.

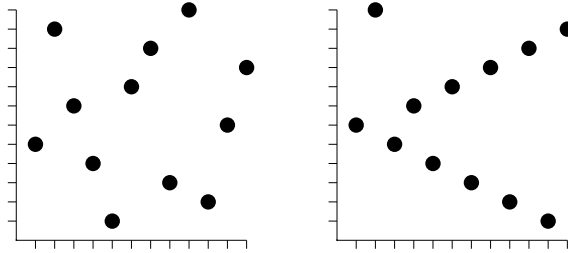


Figure 1. The plots of two simple permutations. Note that every simple subsequence of length at least 4 of the permutation on the right must contain its first two entries.

Each sequence of distinct real numbers is *order isomorphic* to a unique permutation; this is the permutation with the same relative comparisons. We say that a sequence of distinct real numbers is simple if it is order isomorphic to a simple permutation. We prove that long simple permutations must contain two long almost disjoint simple subsequences. Formally:

Theorem 1.1. *There is a function $f(k)$ such that every simple permutation of length at least $f(k)$ contains two simple subsequences, each of length at least k , sharing at most two entries.*

The second “two” in the statement of [Theorem 1.1](#) is best possible, as is demonstrated by the family of simple permutations of the form $m(2m)(m-1)(m+1)(m-2)(m+2)\cdots 1(2m-1)$; the permutation on the right of [Figure 1](#) is of this form. On the other hand, no attempt has been made to optimise the function f ; our proof gives an f of order about k^k . The implications of [Theorem 1.1](#) are discussed in the next section. The proof begins in [Section 3](#).

2. Implications/motivation

The permutation π is said to *contain* the permutation σ , written $\sigma \leq \pi$, if π has a subsequence that is order isomorphic to σ . For example, $\pi = 391867452$ contains $\sigma = 51342$, as can be seen by considering the subsequence $91672 (= \pi(2), \pi(3), \pi(5), \pi(6), \pi(9))$, and such a subsequence is called a *copy* of σ in π . This pattern-containment relation is a partial order on permutations. We refer to downsets of permutations under this order as *permutation classes*. In other words, if \mathcal{C} is a permutation class, $\pi \in \mathcal{C}$, and $\sigma \leq \pi$, then $\sigma \in \mathcal{C}$. We denote by \mathcal{C}_n the set $\mathcal{C} \cap S_n$, i.e. the permutations in \mathcal{C} of length n , and we refer to $\sum |\mathcal{C}_n| x^n$ as the *generating function for \mathcal{C}* . Recall that an *antichain* is a set of pairwise incomparable elements. For any permutation class \mathcal{C} , there is a unique (possibly infinite) antichain B such that $\mathcal{C} = \text{Av}(B) =$

$\{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}$. This antichain B , which consists of the minimal permutations not in \mathcal{C} , is called the *basis* of \mathcal{C} .

In a class with only finitely many simple permutations, long permutations must map nontrivial intervals onto intervals. Thus these classes have a recursive structure in which long permutations are built up from smaller permutations, and so it is natural to expect them to have algebraic generating functions. This is indeed the case:

Theorem 2.1 (Albert and Atkinson [1]). *A permutation class with only finitely many simple permutations has a readily computable algebraic generating function.*

Denote by $\text{Av}(\beta_1^{\leq r_1}, \beta_2^{\leq r_2}, \dots, \beta_k^{\leq r_k})$ the class of permutations that have at most r_1 copies of β_1 , at most r_2 copies of β_2 , and so on².

Theorem 2.2. *If the class $\text{Av}(\beta_1, \beta_2, \dots, \beta_k)$ contains only finitely many simple permutations then for all choices of nonnegative integers r_1, r_2, \dots , and r_k , then the class $\text{Av}(\beta_1^{\leq r_1}, \beta_2^{\leq r_2}, \dots, \beta_k^{\leq r_k})$ also contains only finitely many simple permutations.*

Proof. We wish to show that for any choice of nonnegative integers r_1, r_2, \dots, r_k , only finitely many simple permutations contain at most r_i copies of β_i for each $i \in [k]$. First note that if any β_i is of length 1 or 2 then the theorem follows easily, so we may suppose that $|\beta_i| \geq 3$ for all $i \in [k]$. In the case $\sum r_i = 0$, the conclusion follows from the hypotheses. We now proceed by induction; suppose that $r_j > 0$ and set

$$g(r_1, r_2, \dots, r_k) = f(g(r_1, r_2, \dots, r_{j-1}, \lfloor r_j/2 \rfloor, r_{j+1}, \dots, r_k)),$$

where f denotes the function from Theorem 1.1. By that theorem, every simple permutation π of length at least $g(r_1, r_2, \dots, r_k)$ contains two simple subsequences of length at least $f(g(r_1, r_2, \dots, r_{j-1}, \lfloor r_j/2 \rfloor, r_{j+1}, \dots, r_k))$, and by induction each of these simple subsequences contains more than $\lfloor r_j/2 \rfloor$ copies of β_j . Moreover, because these simple subsequences share at most two entries, their copies of β_j are distinct, and thus π contains more than r_j copies of β_j , proving the theorem. ■

² That this is a permutation class is clear, although finding its basis may be less obvious. An easy argument shows that the basis elements of this class have length at most $\max\{(r_i + 1)|\beta_i| : i \in [k]\}$; see Atkinson [3] for the details. One such computation: $\text{Av}(132^{\leq 1}) = \text{Av}(1243, 1342, 1423, 1432, 2143, 35142, 354162, 461325, 465132)$.

Note that one of the simplest classes with only finitely many simple permutations is $\text{Av}(132)$ ³, and so [Theorem 2.2](#) implies the result of Bóna [5] and Mansour and Vainshtein [17] that for every r , the class of all permutations containing at most r copies of 132 has an algebraic generating function⁴. The largest permutation class whose only simple permutations are 1, 12, and 21 is the class of *separable permutations*, $\text{Av}(2413, 3142)$, and thus as another instance of [Theorem 2.2](#), we have that for all r and s , every subclass of $\text{Av}(2413^{\leq r}, 3142^{\leq s})$ contains only finitely many simple permutations and thus also has an algebraic generating function.

[Theorem 1.1](#) does not apply only to permutation classes. In Brignall, Huczynska, and Vatter [7], [Theorem 2.1](#) is extended to “finite query-complete sets of properties”. As a specialisation of that theorem, we have the following.

Theorem 2.3 (Brignall, Huczynska, and Vatter [7]). *In a permutation class \mathcal{C} with only finitely many simple permutations, the following sequences have algebraic generating functions:*

- the number of even permutations in \mathcal{C}_n ,
- the number of involutions in \mathcal{C}_n ,
- the number of even involutions in \mathcal{C}_n ,
- the number of alternating permutations in \mathcal{C}_n ,
- the number of permutations in \mathcal{C}_n avoiding a finite set of blocked permutations⁵.

There are several results in the literature that follow from the combination of [Theorems 1.1 and 2.3](#):

- Even permutations in $\text{Av}(132^{\leq r})$ – Mansour [16].

(When counting even permutations, unlike when counting all permutations, symmetry considerations reduce us to three cases of length three permuta-

³ In any permutation from $\text{Av}(132)$, all entries to the left of the maximum must be greater than all entries to the right. This shows that $\text{Av}(132)$ has only three simple permutations (1, 12, and 21).

⁴ For example, the generating function in the $r=1$ case is

$$\frac{1 - \sqrt{1 - 4x}}{2x} + \frac{8x^3}{\sqrt{1 - 4x} (1 + \sqrt{1 - 4x})^3}$$

(due, originally, to Bóna [6]).

⁵ *Blocked permutations*, introduced by Babson and Steingrímsson [4], are permutations containing dashes indicating the entries that need not occur consecutively. For example, 51342 contains two copies of 3-12: 513 and 534, but note that 514 is not a copy of 3-12 because the 1 and 4 are not adjacent.

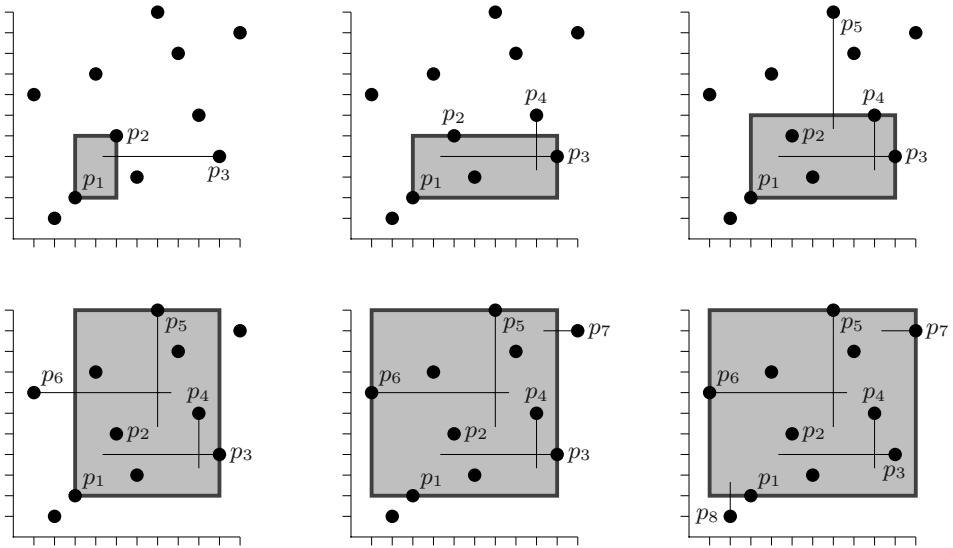


Figure 2. A pin sequence.

tions – 123, 132, and 231 – not two⁶, and thus there is another result we can state: the even permutations in $\text{Av}(231^{\leq r})$ have an algebraic generating function for all r , although this result seems to have escaped print.)

- Involutions in $\text{Av}(231^{\leq r})$ – Mansour, Yan, and Yang [18]. In the same reference: even involutions in $\text{Av}(231^{\leq r})$. The case of involutions in $\text{Av}(132^{\leq r})$ is due to Mansour [private communication].
- Alternating permutations in $\text{Av}(132^{\leq r})$ – Mansour [15].
- Permutations with at most r copies of the blocked permutation 13-2 – Claesson and Mansour [9]⁷.

3. Pin Sequences

Given points p_1, \dots, p_m in the plane, we denote by $\text{rect}(p_1, \dots, p_m)$ the smallest axes-parallel rectangle containing them.

⁶ We have thus far ignored the other case; $\text{Av}(123)$, and thus $\text{Av}(123^{\leq r})$, contains infinitely many simple permutations, so these methods do not apply. The class $\text{Av}(123)$ is enumerated by the Catalan numbers, $\text{Av}(123^{\leq 1})$ was counted by Noonan [19], while $\text{Av}(123^{\leq 2})$ was counted by Fulmek [13], proving a conjecture of Noonan and Zeilberger [20]. No results for larger values are known, although Fulmek conjectures formulas for $r = 3$ and $r = 4$ and further conjectures that $\text{Av}(123^{\leq r})$ has an algebraic generating function for all r .

⁷ While avoiding 132 and avoiding 13-2 are equivalent conditions, a permutation will tend to have fewer copies of 13-2.

Choose two points p_1 and p_2 in the plot of a permutation π . If these two points do not form an interval then there is at least one point which lies outside $\text{rect}(p_1, p_2)$ and slices $\text{rect}(p_1, p_2)$ either horizontally or vertically. (This discussion is accompanied by the sequence of diagrams shown in Figure 2.) We call such a point a *pin*. Choose a pin and label it p_3 . Now consider the larger rectangle $\text{rect}(p_1, p_2, p_3)$. If this also does not form an interval in π then we can find another pin, p_4 , which slices $\text{rect}(p_1, p_2, p_3)$ either horizontally or vertically. Again, if $\text{rect}(p_1, p_2, p_3, p_4)$ is not an interval then we can find another pin p_5 . We refer to a sequence of pins constructed in this manner as a *pin sequence*.

Formally, a pin sequence is a sequence of points p_1, p_2, \dots in the plot of π such that for each $i \geq 3$,

- $p_i \notin \text{rect}(p_1, \dots, p_{i-1})$, and
- if $\text{rect}(p_1, \dots, p_{i-1}) = [a, b] \times [c, d]$ and $p_i = (x, y)$, we have either $a < x < b$ or $c < y < d$, or, in other words, p_i *slices* $\text{rect}(p_1, \dots, p_{i-1})$ either horizontally or vertically.

We describe pins as either *left*, *right*, *up*, or *down* based on their position relative to the rectangle that they slice. Thus in the pin sequence from Figure 2, p_3 and p_7 are right pins, p_4 and p_5 are up pins, p_6 is a left pin, and p_8 is a down pin (p_1 and p_2 lack direction).

A *proper pin sequence* is one that satisfies two additional conditions:

- *Maximality condition*: each pin must be maximal in its direction. For example, if $\text{rect}(p_1, \dots, p_{i-1}) = [a, b] \times [c, d]$ and $p_i = (x, y)$ is a right pin, then it is the right-most of all possible right pins for this rectangle, or, in other words, the region $(x, n] \times [c, d]$ is devoid of points.
- *Separation condition*: p_{i+1} must *separate* p_i from $\{p_1, \dots, p_{i-1}\}$. That is, p_{i+1} must lie horizontally or vertically between $\text{rect}(p_1, \dots, p_{i-1})$ and p_i .

For example, in the pin sequence shown in Figure 2, the choice of p_4 violates the maximality condition, while the choices of p_5 , p_7 , and p_8 violate the separation condition. The ultimate goal of the following succession of lemmas is to show (in Theorem 3.4) that all or all but one of the pins in a proper pin sequence themselves form a simple permutation. We begin by observing that proper pin sequences travel by 90° turns only.

Lemma 3.1. *In a proper pin sequence, p_{i+1} cannot lie in the same or opposite direction as p_i (for all $i \geq 3$).*

Proof. By the maximality condition, p_{i+1} cannot lie in the same direction as p_i . It cannot lie in the opposite direction by the separation condition. ■

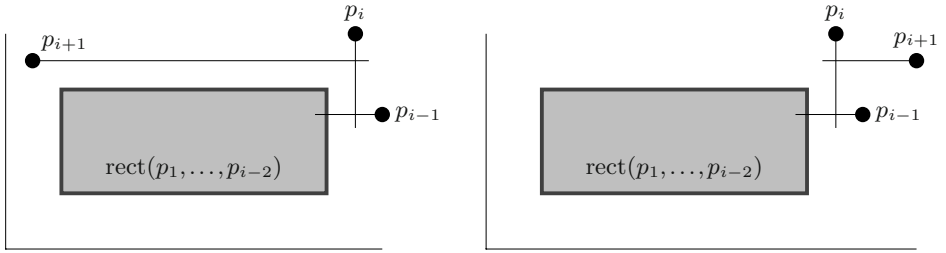


Figure 3. The two cases in the proof of [Lemma 3.3](#).

Lemma 3.2. *In a proper pin sequence, p_i does not separate any two members of $\{p_1, \dots, p_{i-2}\}$.*

Proof. If p_i did separate $\text{rect}(p_1, \dots, p_{i-2})$ into two parts then p_{i-1} would lie on one side of this divide, violating the separation condition. ■

Lemma 3.3. *In a proper pin sequence, p_i and p_{i+1} are separated either by p_{i-1} or by each of p_1, \dots, p_{i-2} .*

Proof. The lemma is vacuously true for $i = 1$ and $i = 2$, so let us assume that $i \geq 3$. Without loss we may assume that p_{i-1} is a right pin and p_i is an up pin. By [Lemma 3.1](#), p_{i+1} must be either a right pin or a left pin. The remainder of the proof is evident from [Figure 3](#). ■

We are now ready to prove our main result about proper pin sequences.

Theorem 3.4. *If p_1, \dots, p_m is a proper pin sequence of length $m \geq 5$ then one of the sets of points $\{p_1, \dots, p_m\}$, $\{p_1, \dots, p_m\} \setminus \{p_1\}$, or $\{p_1, \dots, p_m\} \setminus \{p_2\}$ is order isomorphic to a simple permutation.*

Proof. We are interested in the possible intervals in the subsequence given by the pins p_1, \dots, p_m ; we shall call these *intervals of pins*. The bulk of our proof is devoted to establishing the following claim: for any m , the only possible proper minimal nonsingleton intervals of pins in the proper pin sequence $\{p_1, \dots, p_m\}$ are $\{p_1, p_m\}$, $\{p_2, p_m\}$, $\{p_1, p_3, \dots, p_m\}$ or $\{p_2, \dots, p_m\}$.

Take $M \subseteq \{p_1, \dots, p_m\}$ to be a minimal non-singleton interval of pins. Note that M is therefore order isomorphic to a simple permutation. If M contains a pair of pins p_i and p_j with $i < j < m$ then by the separation condition $p_{j+1}, \dots, p_m \in M$. Furthermore, because $j < m$, [Lemma 3.3](#) shows that M contains either p_{j-1} or p_1, p_2, \dots, p_{j-2} . In the latter case, if $j \geq 4$ then separation gives $p_{j-1} \in M$, as desired, while if $j \leq 3$, we have already found a minimal non-singleton interval of pins of the desired form. In the

former case, the proof is completed by iterating this process. Only the case $M = \{p_i, p_m\}$ remains. If $3 \leq i \leq m-1$ then by the separation condition p_i separates $\{p_1, \dots, p_{i-1}\}$, while [Lemma 3.2](#) shows that p_m does not separate these points; thus at least one of them must lie in M , a contradiction which completes the proof of the claim.

Returning to the proof of the theorem, suppose that $\{p_1, \dots, p_m\}$ is not itself order isomorphic to a simple permutation and that $m \geq 5$. Thus, by the claim, at least one of $\{p_1, p_m\}$, $\{p_2, p_m\}$, $\{p_1, p_3, \dots, p_m\}$ or $\{p_2, \dots, p_m\}$ forms a minimal nonsingleton interval of pins. The latter two cases give us a simple of the desired form, so now assume either $\{p_1, p_m\}$ or $\{p_2, p_m\}$ is an interval of pins. (Note that we cannot have both intervals since p_3 separates p_1 from p_2 .) We assume the former as the latter is analogous. Consider the pin sequence $\{p_2, \dots, p_m\}$. By the claim, the only possible minimal nonsingleton intervals of pins in this sequence are $\{p_2, p_m\}$, $\{p_3, p_m\}$, $\{p_2, p_4, \dots, p_m\}$ or $\{p_3, \dots, p_m\}$. The latter two cases may be ignored since the only interval of pins in the original sequence $\{p_1, \dots, p_m\}$ was $\{p_1, p_m\}$, and hence the points that p_1 separated are the same as those separated by p_m . Thus it remains to eliminate the cases $\{p_2, p_m\}$ and $\{p_3, p_m\}$. Since p_3 separates p_1 from p_2 and $\{p_1, p_m\}$ is an interval, p_3 also separates p_2 from p_m , so $\{p_2, p_m\}$ cannot form an interval of pins for the sequence $\{p_2, \dots, p_m\}$. Similarly, $\{p_3, p_m\}$ cannot be an interval of pins for $\{p_2, \dots, p_m\}$ because p_4 separates p_3 from p_1 and thus also from p_m because we have assumed that $\{p_1, p_m\}$ forms an interval. Thus $\{p_2, \dots, p_m\}$ contains no nontrivial intervals of pins and is therefore order isomorphic to a simple permutation, completing the proof. ■

As a corollary of this theorem, we see that [Theorem 1.1](#) (in fact, a stronger result) is true for simple permutations with long pin sequences.

Corollary 3.5. *If π contains a proper pin sequence of length at least $2k+2$ then π contains two disjoint simple subsequences, each of length at least k .*

Proof. Apply [Theorem 3.4](#) to the two pin sequences p_1, \dots, p_{k+1} and p_{k+2}, \dots, p_{2k+2} . ■

We say that the pin sequence p_1, \dots, p_m for the permutation π of length n is *saturated* if $\text{rect}(p_1, \dots, p_m) = [n] \times [n]$. For example, the pin sequence in [Figure 2](#) is saturated. Any two points $p_1 \neq p_2$ in the plot of a simple permutation can be extended to a saturated pin sequence, as we are forced to stop extending a pin sequence only upon finding an interval or when the rectangle contains every point in π .

It is important to note that two points in a simple permutation need not be extendable to a proper saturated pin sequence. For example, the permutation in [Figure 2](#) does not have a proper saturated pin sequence beginning

with p_1 and p_2 . For this reason we work with a weaker requirement: the pin sequence p_1, \dots, p_m is said to be *right-reaching* if p_m is the right-most point of π .

Lemma 3.6. *For every simple permutation π and pair of points p_1 and p_2 (unless, trivially, p_1 is the right-most point of π), there is a proper right-reaching pin sequence beginning with p_1 and p_2 .*

Proof. Clearly we can find a saturated pin sequence p_1, p_2, \dots in π that satisfies the maximality condition. Since this pin sequence is saturated, it includes the right-most point; label it p_{i_1} . Now take i_2 as small as possible so that $p_1, p_2, \dots, p_{i_2}, p_{i_1}$ is a valid pin sequence. Note first that $i_2 < i_1$ because p_1, \dots, p_{i_1} is a valid pin sequence. Now observe that p_{i_1} separates p_{i_2} from $\text{rect}(p_1, \dots, p_{i_2-1})$, because $p_1, \dots, p_{i_2-1}, p_{i_1}$ is not a valid pin sequence. Continuing in this manner, we find pins p_{i_3}, p_{i_4} , and so on, until we reach the stage where $p_{i_{m+1}} = p_2$. Then $p_1, p_2, p_{i_m}, p_{i_{m-1}}, \dots, p_{i_1}$ is a proper right-reaching pin sequence. ■

4. Simple permutations without long proper pin sequences

It remains only to consider simple permutations without long proper pin sequences. Lemma 3.6 shows that in such a permutation, any two points p_1, p_2 can be extended to a short proper right-reaching pin sequence. Our goal in this section is to use several of these short right-reaching sequences to prove that such permutations contain long “alternations”.

We use the term *horizontal alternation* to refer to a permutation in which every odd entry lies to the left of every even entry, or the reverse of such a permutation. A *vertical alternation* is the group-theoretic inverse of a horizontal alternation. Examples are shown in Figure 4. Every sufficiently

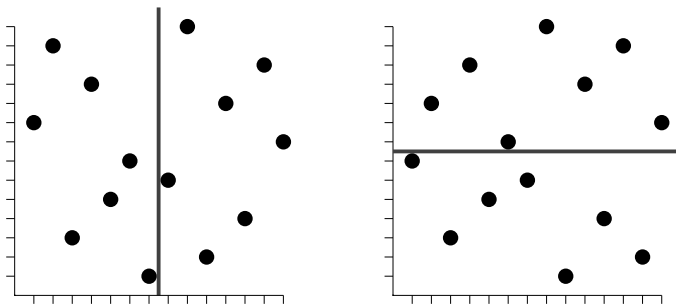


Figure 4. A horizontal alternation (left) and its inverse, a vertical alternation (right).

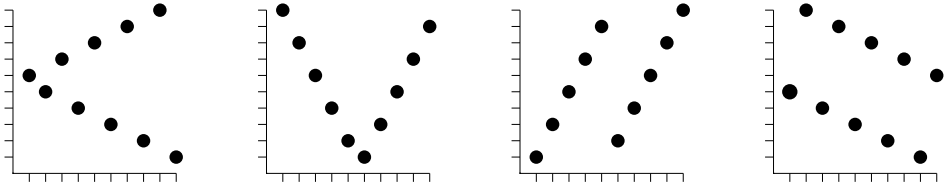


Figure 5. The two permutations on the left are wedge alternations, the two on the right are parallel alternations. Note that we also consider symmetries of such drawings to be alternations.

long alternation contains either a long *parallel alternation* or a long *wedge alternation* (see Figure 5 for definitions):

Proposition 4.1. *Every alternation of length at least $2k^4$ contains either a parallel or wedge alternation of length at least $2k$.*

Proof. Assume by symmetry that π is a vertical alternation of length $2n \geq 2k^4$. By the Erdős–Szekeres Theorem (every permutation of length n contains a monotone subsequence of length at least \sqrt{n}), the sequence $\pi(1), \pi(3), \dots, \pi(2n-1)$ contains a monotone subsequence of length at least k^2 , say $\pi(i_1), \pi(i_2), \dots, \pi(i_{k^2})$. Applying the Erdős–Szekeres Theorem to the subsequence $\pi(i_1+1), \pi(i_2+1), \dots, \pi(i_{k^2}+1)$ completes the proof. ■

Note that every parallel alternation of length $2k+2 \geq 10$ contains two disjoint simple permutations of length at least k . Thus Theorem 1.1 follows in the case where our simple permutation contains a long parallel alternation.

Returning to pin sequences, the pin sequences p_1, \dots and q_1, \dots are said to

- be *initially nonoverlapping* if $\text{rect}(p_1, p_2)$ and $\text{rect}(q_1, q_2)$ are disjoint,
- *converge at the point x* if there exist i and j such that $p_i = q_j = x$ but $\{p_1, \dots, p_{i-1}\}$ and $\{q_1, \dots, q_{j-1}\}$ are disjoint.

A collection of pin sequences converges or is initially nonoverlapping if they pairwise converge or are pairwise initially nonoverlapping. Note that it is always possible to find a collection of $\lfloor n/2 \rfloor$ initially nonoverlapping proper pin sequences in a permutation π of length n by taking proper pin sequences beginning with the first and second points, the third and fourth points, and so on, reading left to right.

Lemma 4.2. *If $16k$ initially nonoverlapping proper pin sequences of π converge at the same point, then π contains an alternation of length at least $2k$.*

Proof. Suppose that $16k$ initially nonoverlapping proper pin sequences converge at the point x . Note that x can be the first or second pin for at most

one of these sequences because they are initially nonoverlapping. Thus one of the following two possibilities must occur:

- at least $8k$ of the sequences have x as their third pin, or
- at least $8k$ of the sequences have x as their fourth or later pin.

Suppose that at least $8k$ of the sequences have x as their third pin. This point could be variously functioning as a left, right, down, or up pin for each of these $8k$ sequences, but x plays the same role for at least $2k$ sequences. Suppose, by symmetry, that x is a right pin for at least $2k$ sequences. Since x is the third pin for these sequences, one of their first two pins lies above x while the other lies below and because these sequences are initially nonoverlapping, an alternation of length at least $2k$ can be obtained by choosing one point from each sequence.

Now suppose that at least $8k$ of the sequences have x as their fourth or later pin. Again we may assume without loss that x is a right pin for at least $2k$ of these sequences. Now consider the immediate predecessors to x in these sequences. These pins are either up pins or down pins (by [Lemma 3.1](#)). By symmetry we may assume that for at least k of these sequences the immediate predecessor to x is an up pin. Reading left to right, label these immediate predecessor pins $p^{(1)}, p^{(2)}, \dots, p^{(k)}$ and let $R^{(i)}$ denote the rectangle for which $p^{(i)}$ is a pin. Note that each $R^{(i)}$ lies completely below x , as otherwise the separation condition would prevent x from following $p^{(i)}$ in the corresponding pin sequence. We now have the situation depicted in [Figure 6](#).

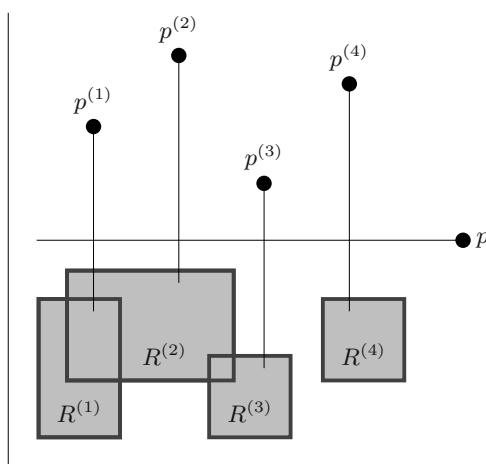


Figure 6. The situation that arises in the proof of [Lemma 4.2](#).

It suffices to show, for each i , that π contains a point lying horizontally between $p^{(i)}$ and $p^{(i+1)}$ and below x since these points, together with the $p^{(i)}$'s and x , will give an alternation of length $2k$. However, if there is no such point then $p^{(i)}$ and $p^{(i+1)}$ could each function as up pins for both $R^{(i)}$ and $R^{(i+1)}$, and thus one of these choices would contradict the maximality condition, completing the proof. ■

Lemma 4.3. *Every simple permutation of length at least $2(16k^4)^{2k}$ contains either a proper pin sequence of length at least $2k$ or a parallel or wedge alternation of length at least $2k$.*

Proof. Suppose that the simple permutation π of length n contains neither a proper pin sequence of length at least $2k$ nor a parallel or wedge alternation of length at least $2k$. In particular, then, π does not contain a proper right-reaching pin sequence of length $2k$, and it follows from Proposition 4.1 that π has no alternations of length $2k^4$.

It follows from our earlier comments that π contains a collection of $\lfloor n/2 \rfloor$ initially nonoverlapping proper right-reaching pin sequences. As these sequences are right-reaching, they all have the same final (right-most) pin which we denote by p . By Lemma 4.2, fewer than $16k^4$ of these pin sequences converge at p ; equivalently, there are fewer than $16k^4$ distinct immediate predecessors to p . Label these immediate predecessors $p^{(1)}, p^{(2)}, \dots, p^{(m)}$. Again, fewer than $16k^4$ pin sequences converge at each of the $p^{(i)}$'s, so there are fewer than $(16k^4)^2$ immediate predecessors to these pins. Continue this process until we reach the sequences of length $2k$, of which we have assumed there are none. We have thus counted all $\lfloor n/2 \rfloor$ of our sequences, and have obtained the bound

$$\lfloor n/2 \rfloor < 1 + 16k^4 + (16k^4)^2 + (16k^4)^3 + \dots + (16k^4)^{(2k-1)},$$

so, simplifying,

$$n < 2(16k^4)^{2k}. \quad \blacksquare$$

We are left to deal with simple permutations which do not have long proper pin sequences but do have long wedge alternations. We prove that these permutations contain long *wedge simple permutations*, of which there are two types (up to symmetry). Examples of these two types are shown in Figure 7.

Lemma 4.4. *If a simple permutation contains a wedge alternation of length $4k^2$ then it contains either a pin sequence of length at least $2k$ or a wedge simple permutation of length at least $2k$.*

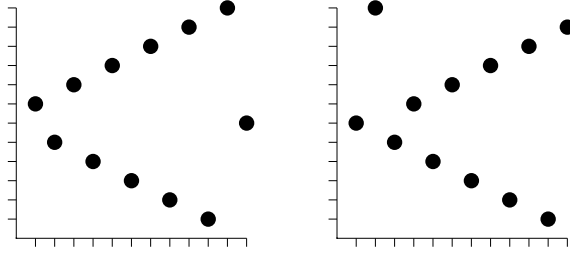


Figure 7. The two types of wedge simple permutations, type 1 (left) and type 2 (right).

Proof. Let π be a simple permutation containing a wedge alternation of length at least $4k^2$. By symmetry we may assume that this wedge alternation opens to the right (i.e., it is oriented as $<$). We call these the *wedge points* of π . Label the two left-most wedge points p_1 and p_2 and by [Lemma 3.6](#) extend this into a proper right-reaching pin sequence p_1, p_2, \dots, p_m .

Let R_i denote the smallest rectangle in the plot of π containing p_1, p_2 , and p_i that is not sliced by a wedge point outside the rectangle. Define the *wedge sum* of the pin p_i , $ws(p_i)$, to be the number of wedge points in R_i . For $i \geq 2$ define the *wedge contribution* of p_i by $wc(p_i) = ws(p_i) - ws(p_{i-1})$ and set $wc(p_1) = 1$. Regarding these quantities we make four observations:

- (W1) the wedge sum of p_m is equal to the total number of wedge points and also to $wc(p_1) + wc(p_2) + \dots + wc(p_m)$;
- (W2) it is not hard to construct examples in which pins have negative wedge contributions; indeed,
- (W3) left pins cannot have positive wedge contributions; and finally,
- (W4) if p_i is an up pin, then the right-most wedge point in R_i is an upper wedge point.

We now claim that each p_i lies in a wedge simple permutation of length at least $wc(p_i) + 2$. This claim implies the theorem, because if no pin lies in a wedge simple permutation of length at least $2k$ then $wc(p_i) \leq 2k - 3$, so by (W1),

$$4k^2 \leq \sum_{i=1}^m wc(p_i) \leq m(2k - 3),$$

and thus $m \geq 2k$, giving the long pin sequence desired.

The claim is easily observed for $i = 1$ and, by (W3), vacuously true if p_i is a left pin. Thus by symmetry there are only three cases to consider: an up pin followed by a right pin, a right pin followed by an up pin, and a left pin followed by an up pin. These three cases are depicted in [Figure 8](#).

Let us consider in detail the case of an up pin followed by a right pin. By (W4), the left-most wedge point in $R_i \setminus R_{i-1}$ lies below p_1 . By separation,

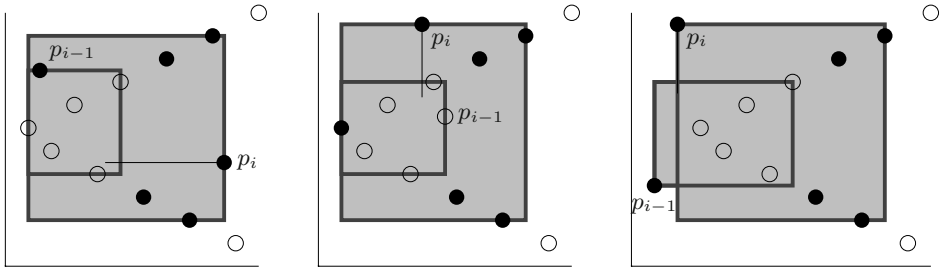


Figure 8. The three cases in the proof of [Lemma 4.4](#); the solid points form simple permutations.

p_{i-1} lies above p_i , which is itself the right-most point in R_i . Therefore the wedge points in $R_i \setminus R_{i-1}$ together with p_i and p_{i-1} constitute a type 1 wedge simple permutation. The other cases follow by similar analysis; in the right-up case the wedge points in $R_i \setminus R_{i-1}$ together with p_1 and p_i give a wedge simple permutation of type 2, while in the left-up case a wedge simple permutation of type 2 can be formed from the wedge points in $R_i \setminus R_{i-1}$, p_{i-1} , and p_i . ■

We have therefore established the following theorem.

Theorem 4.5. *Every simple permutation of length at least $2(256k^8)^{2k}$ contains a proper pin sequence of length $2k$, a parallel alternation of length $2k$, or a wedge simple permutation of length $2k$.*

The proof of [Theorem 1.1](#) now follows by analysing each of these cases in turn. A parallel alternation of length $2k + 2 \geq 10$ contains two disjoint simple permutations of length k . A type 1 wedge simple permutation of length $2k$ contains two type 1 wedge simple permutations of length k with only one entry in common, and a type 2 wedge simple permutation of length $2k$ contains two type 2 wedge simple permutations of length k which share two entries. Finally, [Corollary 3.5](#) shows that a permutation with a proper pin sequence of length $2k + 2$ contains two disjoint simple permutations of length k .

Brignall, Ruškuc, and Vatter [8] apply [Theorem 4.5](#) to show that it is possible to decide whether or not a permutation class contains only finitely many simple permutations, and expatiate upon [Lemma 4.3](#), showing that every long simple permutation contains either a long alternation or a long “oscillation”.

5. Other contexts

Although our proof is highly permutation-centric, there is no reason why analogues of [Theorem 1.1](#) cannot exist for other types of object. For example, an interval⁸ in a graph is a set of vertices $X \subseteq V(G)$ such that $N(v) \setminus X = N(w) \setminus X$ for all $v, w \in X$, where $N(v)$ denotes the neighbourhood of v in G . A graph on n vertices therefore has several trivial intervals (\emptyset , $V(G)$, and the singletons); a graph with no nontrivial intervals is then often called *prime* or *indecomposable* (the word simple meaning something completely different in this context). These graphs have been the subject of considerable study, see Ehrenfeucht, Harju, and Rozenberg [11], Ille [14], and Sabidussi [21].

The most general context for simplicity – and thus the most general context for results such as [Theorem 1.1](#) – is relational structures. Let \mathcal{L} denote a relational language (i.e., a set of relational symbols together with positive integers n_R for each relational symbol $R \in \mathcal{L}$, specifying the arity of R) and \mathcal{A} an \mathcal{L} -structure (i.e., a ground set $\text{dom}(\mathcal{A})$ together with interpretations of the relational symbols from \mathcal{L}). Following Földes [12], we say that the subset $X \subseteq \text{dom}(\mathcal{A})$ is an interval if the following occurs for every relation $R \in \mathcal{L}$ and every n_R -tuple $(x_1, x_2, \dots, x_{n_R}) \in \text{dom}(\mathcal{A})^{n_R} \setminus X^{n_R}$: if $x_i \in X$ then the value of $R^{\mathcal{A}}(x_1, x_2, \dots, x_{n_R})$ is unchanged by swapping x_i with any other element of X . Again the relational structure \mathcal{A} will have the trivial intervals \emptyset , $\{a\}$ for all $a \in \text{dom}(\mathcal{A})$, and $\text{dom}(\mathcal{A})$ itself, and it is simple if it has no others.

In this most general context, any analogue of [Theorem 1.1](#) would need to allow for more intersection between the two simple substructures. An example demonstrating this is given in Footnote 9.

References

- [1] M. H. ALBERT and M. D. ATKINSON: Simple permutations and pattern restricted permutations, *Discrete Math.* **300**(1–3) (2005), 1–15.
- [2] M. H. ALBERT, M. D. ATKINSON and M. KLAZAR: The enumeration of simple permutations, *J. Integer Seq.* **6**(4) (2003), Article 03.4.4, 18 pp. (electronic).
- [3] M. D. ATKINSON: Restricted permutations, *Discrete Math.* **195**(1–3) (1999), 27–38.

⁸ These are also called autonomous sets, blocks, bound sets, clans, closed sets, clumps, committees, congruences, convex sets, externally related sets, factors, modules, partie solidaires, partive sets, stable sets, and strong intervals.

⁹ Let \mathcal{L} consist of a 2-ary relation $<$ and a k -ary relation R . Take \mathcal{A} with $\text{dom}(\mathcal{A}) = [2n]$ where $<$ is interpreted as the normal linear order on $[2n]$ and $R(1, 3, 5, \dots, 2k-3, i)$ precisely for even $i \in [2k-2, 2n]$. This structure is simple, but all simple substructures (with at least two elements) of \mathcal{A} must contain each of $1, 3, 5, \dots, 2k-3$, and then to prevent these elements from containing a nontrivial interval, the simple substructure must also contain $2, 4, 6, \dots, 2k-4$.

- [4] E. BABSON and E. STEINGRÍMSSON: Generalized permutation patterns and a classification of the Mahonian statistics, *Sém. Lothar. Combin.* **44** (2000), Art. B44b, 18 pp. (electronic).
- [5] M. BÓNA: The number of permutations with exactly r 132-subsequences is P -recursive in the size!, *Adv. in Appl. Math.* **18**(4) (1997), 510–522.
- [6] M. BÓNA: Permutations with one or two 132-subsequences, *Discrete Math.* **181**(1–3) (1998), 267–274.
- [7] R. BRIGNALL, S. HUCZYNSKA and V. VATTER: Simple permutations and algebraic generating functions, *J. Combin. Theory Ser. A* **115**(3) (2008), 423–441.
- [8] R. BRIGNALL, N. RUŠKUC and V. VATTER: Simple permutations: Decidability and unavoidable substructures, *Theor. Comp. Sci.* **391**(1–2) (2008), 150–163.
- [9] A. CLAESSON and T. MANSOUR: Counting occurrences of a pattern of type $(1, 2)$ or $(2, 1)$ in permutations, *Adv. in Appl. Math.* **29**(2) (2002), 293–310.
- [10] S. CORTEEL, G. LOUCHARD and R. PEMANTLE: Common intervals of permutations, in *Mathematics and computer science, III*, Trends Math., Birkhäuser, Basel, 2004, pp. 3–14.
- [11] A. EHRENFUCHT, T. HARJU and G. ROZENBERG: *The theory of 2-structures*, World Scientific Publishing Co. Inc., River Edge, NJ, 1999.
- [12] S. FÖLDES: On intervals in relational structures, *Z. Math. Logik Grundlag. Math.* **26**(2) (1980), 97–101.
- [13] M. FULMEK: Enumeration of permutations containing a prescribed number of occurrences of a pattern of length three, *Adv. in Appl. Math.* **30**(4) (2003), 607–632.
- [14] P. ILLE: Indecomposable graphs, *Discrete Math.* **173**(1–3) (1997), 71–78.
- [15] T. MANSOUR: Restricted 132-alternating permutations and Chebyshev polynomials, *Ann. Comb.* **7**(2) (2003), 201–227.
- [16] T. MANSOUR: Counting occurrences of 132 in an even permutation, *Int. J. Math. Math. Sci.* **25**–**28** (2004), 1329–1341.
- [17] T. MANSOUR and A. VAINSHTEIN: Counting occurrences of 132 in a permutation, *Adv. in Appl. Math.* **28**(2) (2002), 185–195.
- [18] T. MANSOUR, S. H. F. YAN and L. L. M. YANG: Counting occurrences of 231 in an involution, *Discrete Math.* **306**(6) (2006), 564–572.
- [19] J. NOONAN: The number of permutations containing exactly one increasing subsequence of length three, *Discrete Math.* **152**(1–3) (1996), 307–313.
- [20] J. NOONAN and D. ZEILBERGER: The enumeration of permutations with a prescribed number of “forbidden” patterns, *Adv. in Appl. Math.* **17**(4) (1996), 381–407.
- [21] G. SABIDUSSI: Graph derivatives, *Math. Z.* **76** (1961), 385–401.

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